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J M HERMANN ET AL. SEP 87 MM-5488-87-10

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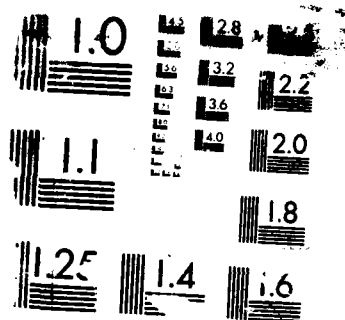
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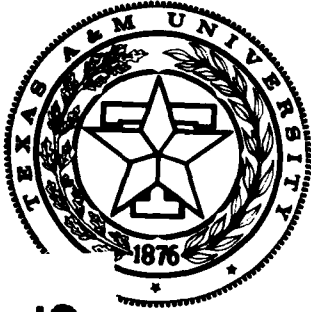
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ON THE ENERGY RELEASE RATE FOR DYNAMIC  
TRANSIENT ANTI-PLANE SHEAR CRACK PROPA-  
GATION IN A GENERAL LINEAR VISCOELASTIC  
BODY

J.M. HERRMANN  
AND  
J.R. WALTON

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On the Energy Release Rate for Dynamic Transient  
Anti-Plane Shear Crack Propagation  
in a General Linear Viscoelastic Body

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Summary.

The problem of a semi-infinite mode III crack that suddenly begins to propagate at a constant speed is considered for a general linear viscoelastic body. A simple closed form expression for the Laplace transform of the energy release rate (ERR) is derived under the assumption that a Barenbatt type failure zone exists at the crack tip. The first two terms of a short time asymptotic series for the ERR is constructed and the rate at which the ERR converges to steady-state is studied. It is shown that the rate of convergence to steady-state is dependent upon crack speed and material properties. Moreover, it is found that whether or not a failure zone is incorporated into the model significantly influences both quantitatively and qualitatively the short and long time behavior of the ERR. This difference is important to predictions of stable vs unstable crack speeds based upon a critical ERR fracture criterion.

## 1. Introduction.

Several analytical studies of dynamically propagating cracks in linear viscoelastic material have appeared in the literature since Willis(1967) presented an analysis of the dynamic, steady-state propagation of a semi-infinite, mode III (anti-plane shear) crack in an infinite viscoelastic body. Employing transform methods and the Wiener-Hopf technique, Willis constructed the dynamic stress intensity factor (SIF) for a special class of crack face loadings and for a standard linear solid material model. Subsequently, Atkinson and List(1972) introduced transient effects into the problem by assuming that the crack, initially at rest, begins to propagate at a constant speed under the action of suddenly applied loads on the crack faces. Also utilizing the Wiener-Hopf method, they derived an expression for the Laplace transform of the time dependent SIF from which long and short time asymptotic approximations and numerical Laplace inversion calculations were obtained. However, their analysis was limited to consideration of constant applied load along the crack faces and the required Wiener-Hopf factorization was effected only for a Maxwell material model and the Achenbach-Chao(1962) three parameter approximation to the standard linear solid.

Somewhat later, Atkinson and Coleman(1977) used a

matched asymptotic expansion technique to develop an approximate analysis of the steady-state propagation of a semi-infinite mode I (plane strain or plane stress) crack propagating in a clamped viscoelastic strip. Shortly thereafter, Atkinson(1979) presented an approximate analysis of the mode I counterpart to the mode III problem considered by Atkinson and List(1972). Their argument, ostensibly valid for fairly general material models, involved slightly modifying the exact elastic result of Baker(1962) in order to approximate the Laplace transform of the actual dynamic viscoelastic SIF. The dominant term for each of the short and long time asymptotic expansions of this approximate SIF was derived for each of three different applied crack face loads: a constant, a delta function point force and an exponentially decreasing form. Also in that paper, Atkinson reconsidered the mode III problem and extended the Atkinson and List analysis to handle the above three types of crack face loadings. However, consideration was limited again to the Achenbach-Chao material model. Atkinson also constructed an expression for the energy release rate (ERR) based upon a local (i.e. at the crack tip) work argument and the singular stress field.

Also in that same year, Atkinson and Popelar(1979) presented an analysis of the transient constant crack speed mode III problem for a viscoelastic strip. Constitutive



relations in terms of differential operators were assumed and the external load consisted either of constant displacement of the upper and lower layer boundaries or constant tractions on them. The crack faces were assumed to be stress free. Again the Wiener-Hopf method was used to construct an exact expression for the Laplace transform of the SIF. The required Wiener-Hopf factorization was carried out modulo a term involving a Cauchy type integral. Atkinson and Popelar then restricted attention to numerically approximating the Cauchy integral for the steady-state limit case and assumed a standard linear solid material model.

A year later, Atkinson and Popelar(1980) addressed the more difficult mode I problem for a viscoelastic strip. Again by use of the Wiener-Hopf method, a formal expression for the Laplace transform of the SIF was constructed containing a complicated Cauchy integral. As with the corresponding mode III case, the integral was studied numerically in the limiting special case of steady-state crack propagation in a standard linear solid.

Somewhat later, Walton(1982) examined further the steady-state mode III problem considered by Willis(1967). Utilizing the Riemann-Hilbert rather than the Wiener-Hopf methodology, he constructed a simple closed form expression for the SIF valid for general crack face loadings and very

general material models. More specifically, constitutive equations expressed in terms of convolution integrals rather than differential operators were adopted and the results were shown to be valid irrespective of any assumed time rate of decay of the viscoelastic shear modulus. In contrast, constitutive relations in terms of constant coefficient differential operators, necessarily force an exponentially decaying modulus thereby preventing consideration of the important class of power-law models which more effectively represent the mechanical response of many real viscoelastic materials, such as rubber, than do exponentially decaying functions.

Subsequently, Walton(1983) extended the above analysis to determine the angular dependence of the stress field in a neighborhood of the crack tip. In particular, it was shown that the asymptotic stress field at the crack tip has the same angular dependence as the corresponding dynamic elastic problem. Only the SIF differs between the elastic and viscoelastic fields.

Walton(1985) next considered the steady-state mode III problem for a viscoelastic strip. Again utilizing the Riemann-Hilbert method, a closed form expression for the SIF was constructed for general loadings and shear modulus. The form of the solution exhibits clearly the combined effects of material properties, crack speed and layer thickness upon

the SIF.

More recently, Walton(1987a) reconsidered steady-state mode III crack propagation in an infinite viscoelastic body in order to investigate the implications of including a failure zone model of Barenblatt(1962) type into the determination of the ERR from a global energy balance calculation. A simple closed form expression for the ERR, which in this case is just the work done by the tractions in the failure zone, was derived under the same mild conditions on the shear modulus assumed in Walton(1982) and for a fairly broad class of crack face and failure zone loadings. It was then observed that whether or not a failure zone is incorporated into the model greatly influences both qualitatively and quantitatively the dependence of the ERR upon crack speed and material properties. In particular, calculations based upon a failure zone seem to reflect more closely experimental observations of cracks rapidly propagating in real viscoelastic material.<sup>1</sup>

The methods of Walton(1987a) can be applied to the calculation of the ERR for a wide variety of dynamic viscoelastic crack problems. Schovanec and Walton(1987c) recently completed the analysis of the dynamic steady-state propagation of two parallel mode III cracks in an infinite

<sup>1</sup> Private communication with Prof. W.G. Knauss and Prof. K. Kuo.

viscoelastic body. Also, Walton(1987b) has recently completed a study of the mode I analog of Walton(1987a). In both of these investigations a Barenblatt failure zone model was adopted. Of related interest are two additional papers by Schovanec and Walton(1987a,1987b) in which these same methods were applied to quasi-static mode I crack propagation in non-homogeneous viscoelastic material. It should also be noted that Knauss(1973) and Schapery(1975) applied the Barenblatt model to quasi-static viscoelastic crack growth in viscoelastic material and observed that whether or not a failure zone is incorporated into the model greatly affects the behavior of the ERR.

The present paper applies the above Barenblatt failure zone/Riemann-Hilbert program to the transient mode III problem considered by Atkinson and List(1972). In particular a simple closed form expression for the Laplace transform of the ERR is derived for a very general class of material models. Only crack face loadings of a simple exponentially decaying type are considered but the extension to more general loadings, described in Walton(1987a), offers no difficulty in the present context and is omitted for the sake of brevity. Moreover, the loadings assumed here contain all of the essential ingredients of the Barenblatt model and as such are amply suited to illustrating the qualitative features expected from more general mathematical

models.

From the Laplace transform of the ERR both long time and short time asymptotic analyses are considered. In particular, the first two terms of a short time asymptotic series is derived and the question of the convergence rate to the steady-state solution is addressed. It is shown that the rate of convergence to the steady-state solution is greatly dependent upon crack speed and material properties. Moreover, whether or not a failure zone is assumed to exist greatly influences both the short time and long time behavior of the ERR as well as predictions of stable vs unstable crack speeds based upon a critical ERR fracture criterion.

## 2. Problem Formulation and Stress Analysis.

The problem to be considered is that of a semi-infinite mode III crack that begins to propagate at a constant speed  $v$  in an infinite viscoelastic body due to the sudden application of crack face tractions that then travel with the crack. The governing field equations for the motion of a linear viscoelastic solid are

$$\rho \ddot{u}_i = \sigma_{ij,j}, \quad \epsilon_{ij} = (u_{i,j} + u_{j,i})/2, \quad \sigma_{ij} = 2\mu * d\epsilon_{ij} + \delta_{ij} \lambda * d\epsilon_{kk}, \quad (2.1)$$

where  $\sigma_{ij}$ ,  $\epsilon_{ij}$ , and  $u_i$  denote the stress, strain, and displacement fields respectively. In (2.1),  $\mu * d\epsilon$  denotes the Riemann-Stieltjes convolution  $\mu * d\epsilon = \int_{-\infty}^t \mu(t-\tau) d\epsilon(\tau)$ .

Since the deformation is assumed to be antiplane strain,  $u_1=0$ ,  $u_2=0$ , and the only equation of motion not identically satisfied is  $\mu * d\Delta u = \rho \ddot{u}$ .

A semi-infinite crack lying along the negative  $x_1$ -axis is assumed to begin to propagate at time  $t=0$  with a constant speed  $v$  driven by loads  $\sigma_2, (x_1, 0, t) = f(x_1 - vt)$  which follow it. The corresponding initial-boundary value problem is

$$\rho \ddot{u} = \mu * d\Delta u = \Delta \left[ \mu(0) u, (x_1, x_2, t) + \int_0^t u, (x_1, x_2, \tau) \mu'(t-\tau) d\tau \right] \quad (2.2)$$

$$\text{with initial conditions } u, = 0, \dot{u}, = 0 \text{ at } t=0 \quad (2.3)$$

$$\text{and boundary conditions } \sigma_2, (x_1, 0, t) = f(x_1 - vt) \quad x_1 < vt \quad (2.4)$$

$$u, (x_1, 0, t) = 0 \quad x_1 > vt$$

$$\sigma_2, (x_1, x_2, t) \rightarrow 0 \text{ as } x_1^2 + x_2^2 \rightarrow \infty.$$

From (2.1) it can be seen that

$$\sigma_{2,2}(x,y,t) = \frac{\partial}{\partial x_2} (\mu * u_2) \\ = \frac{\partial}{\partial x_2} \left[ \mu(0)u_2(x_1, x_2, t) + \int_0^t u_2(x_1, x_2, \tau) \mu'(t-\tau) d\tau \right]. \quad (2.5)$$

It is convenient to change from the fixed coordinates  $(x_1, x_2, t)$  to the moving coordinate system  $(x, y, t)$  given by  $x = x_1 - vt$ ,  $y = x_2$  and to define  $w(x, y, t)$  by  $w(x, y, t) = w(x_1 - vt, x_2, t) = u_2(x_1, x_2, t)$ . In the moving coordinates, equation (2.2) becomes

$$\rho \left[ \frac{\partial}{\partial t} - v \frac{\partial}{\partial x} \right]^2 w(x, y, t) = \Delta \left[ \mu(0)w(x, y, t) + \int_0^t w(x_1 - v\tau, x_2, \tau) \mu'(t-\tau) d\tau \right] \\ = \Delta \left[ \mu(0)w(x, y, t) + \int_0^t w(x + vt - v\tau, y, \tau) \mu'(t-\tau) d\tau \right]. \quad (2.6)$$

Application of the Fourier transform, defined by  $\hat{f}(p, y, t) = \int_{-\infty}^{\infty} e^{ipx} f(x, y, t) dx$ , to equation (2.6) results in

$$\rho \left[ \frac{\partial}{\partial t} + ivp \right]^2 \hat{w}(p, y, t) = \left[ (-ip)^2 + \frac{\partial^2}{\partial y^2} \right] \left[ \mu(0)\hat{w}(p, y, t) + \int_0^t \hat{w}(p, y, \tau) e^{-ivp(t-\tau)} \mu'(t-\tau) d\tau \right]. \quad (2.7)$$

Subsequent application of the Laplace transform

$$\bar{g}(p, y, s) = \int_0^{\infty} g(p, y, \tau) e^{-s\tau} d\tau \text{ to (2.7) yields the equation} \\ \rho(s + ivp)^2 \bar{w}(p, y, s) = \left[ \frac{\partial^2}{\partial y^2} - p^2 \right] \left[ \tilde{\mu}(s + ivp) \bar{w}(p, y, s) \right] \quad (2.8)$$

in which  $\tilde{\mu}(s)$  is the Carson transform of the shear modulus

$$\text{given by } \tilde{\mu}(s) = s\bar{\mu}(s) = \mu(0) + \int_0^{\infty} e^{-s\tau} d\mu.$$

Equation (2.8) can be rewritten as

$$\frac{\partial^2}{\partial y^2} \bar{w} - \left[ p^2 + \frac{\rho}{\tilde{\mu}(s+ivp)} (s+ivp)^2 \right] \bar{w} = 0 \text{ which has the solution}$$

$$\bar{w}(p, y, s) = A(p, s) e^{-\beta(s, p) |y|} \quad (2.9)$$

where  $\beta(s, p) = \left[ p^2 + \frac{\rho}{\tilde{\mu}(s+ivp)} (s+ivp)^2 \right]^{(1/2)}$  must be chosen so that  $\operatorname{Re} \beta \geq 0$ .

In a similar manner, the Fourier and Laplace transforms may be applied to the constitutive equation (2.5) to produce

$$\bar{\sigma}_2(p, y, s) = \tilde{\mu}(s+ivp) \frac{\partial}{\partial y} \bar{w}(p, y, s). \quad (2.10)$$

If one defines  $\hat{f}^+(p)$  and  $\hat{f}^-(p)$  by  $\hat{f}^+(p) = \int_0^\infty e^{ipx} f(x) dx$  and

$\hat{f}^-(p) = \int_{-\infty}^0 e^{ipx} f(x) dx$ , then the boundary condition (2.4)

transforms to

$$\bar{\sigma}_2^+(p, 0, s) + \bar{\sigma}_2^-(p, 0, s) = \tilde{\mu}(s+ivp) \frac{\partial}{\partial y} \bar{w}(p, 0, s)$$

$$= -\beta(s, p) \tilde{\mu}(s+ivp) \bar{w}(p, 0, s). \quad (2.11)$$

From (2.4) it can be seen that  $\bar{\sigma}_2^-(p, 0, s) = \bar{f}(p, s)$  and  $\bar{w}(p, 0, s) = \bar{w}^-(p, 0, s)$ . It is assumed a priori (and is easily verified a posteriori) that  $\bar{w}^-(p, 0, s)$  and  $\bar{\sigma}_2^+(p, 0, s)$  have analytic extensions  $\bar{w}^-(z, 0, s)$  and  $\bar{\sigma}_2^+(z, 0, s)$  for  $\operatorname{Im}(z) < 0$  and  $\operatorname{Im}(z) > 0$ , respectively, which vanish as  $|z| \rightarrow \infty$ . Thus the transformed boundary condition (2.11) can be recast as the Riemann-Hilbert problem: find  $F^+(z)$  analytic for  $\operatorname{Im}(z) > 0$  and  $F^-(z)$  analytic for  $\operatorname{Im}(z) < 0$  such that

$$\lim_{\operatorname{Im}(z) \rightarrow +\infty} F^+(z) = \lim_{\operatorname{Im}(z) \rightarrow -\infty} F^-(z) = 0 \text{ and on } \operatorname{Im}(z) = 0,$$



$$F^+(p) = T(p)F^-(p) - \bar{f}(p, s) \quad \text{for } p \in (-\infty, \infty) \quad (2.12)$$

where  $F^+(z) = \bar{\sigma}_z^+(z, 0, s)$ ,  $F^-(z) = \bar{w}^-(z, 0, s)$ , and

$$T(p) = -\tilde{\mu}(s+ivp) \left[ p^2 + \frac{\rho}{\tilde{\mu}(s+ivp)} (s+ivp)^2 \right]^{(1/2)}.$$

It is well known that the solution of (2.12) is

$$F^\pm(z) = X^\pm(z) \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{-\bar{f}(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - z}, \quad (2.13)$$

where  $X^\pm(z)$  solves the homogeneous Riemann-Hilbert problem

$$X^+(p) = T(p)X^-(p) \quad \text{for } p \in (-\infty, \infty). \quad (2.14)$$

To solve (2.14), it is convenient to factor  $T(p)$  into the product

$$T(p) = G_1(p)G_2(p)G_3(p) \quad (2.15)$$

in which  $G_1(p) = -\tilde{\mu}(s+ivp)$ ,  $G_2(p) = \left[ (p - \frac{is}{v})^2 \right]^{(1/2)}$ , and

$$G_3(p) = \left[ \frac{p^2}{(p - \frac{is}{v})^2} - \frac{\rho v^2}{\tilde{\mu}(s+ivp)} \right]^{(1/2)}. \quad X^\pm(z) \text{ may now be}$$

constructed as the product  $X^\pm(z) = X_1^\pm(z)X_2^\pm(z)X_3^\pm(z)$  with

each  $X_i^\pm(z)$  satisfying the Riemann-Hilbert problem

$$X_i^+(p) = G_i(p)X_i^-(p).$$

What will ultimately be required is to solve the homogeneous Riemann-Hilbert problem (2.14) for each fixed  $s$  on a Bromwich path with  $\text{Re}(s) > 0$ . This will be accomplished by first assuming  $s$  real and positive and then invoking an analytic continuation argument. Additionally, for the subsequent analysis the shear modulus will be assumed to be positive, continuously differentiable, non-increasing,

convex and such that  $\mu(\infty) = \lim_{t \rightarrow \infty} \mu(t) > 0$ . Convexity is sufficient but certainly not necessary to insure the validity of the following calculations and though theoretically overly restrictive, it holds for most of the customary models such as a standard linear solid or a power-law material. Moreover, it is worth noting that no explicit time decay rate for the shear modulus needs to be specified for the results to be valid. From the fact that  $\mu(t) = 0$  for  $t < 0$ , it easily follows that  $[\tilde{\mu}(s+ivz)]^{-1}$  is analytic for  $\text{Im}(z) < 0$ . Therefore one may choose  $X_1^+(z) = 1$  and  $X_1^-(z) = -[\tilde{\mu}(s+ivp)]^{-1}$ . (2.16)

Since the product  $G_2(p)G_3(p) = \beta(s, p)$ , the branches of  $G_2(p)$  and  $G_3(p)$  must be chosen so that their product satisfies the requirement that  $\text{Re } \beta(s, p) \geq 0$ . This condition can be met by choosing the branch of  $z^{(1/2)}$  with branch cut along the negative real axis for both  $G_2(p)$  and  $G_3(p)$ . (See

figure 2.1.) Therefore  $G_2(p)$  can be expressed as

$$G_2(p) = \text{sgn}(p) \left(p - \frac{is}{v}\right) \text{ and } X_2^\pm(z) \text{ may be chosen to be } X_2^+(z) = \omega^+(z) \text{ and } X_2^-(z) = \omega^-(z) \left(p - \frac{is}{v}\right)^{-1} \quad (2.17)$$

in which  $\omega^+(z) = z^{(1/2)}$  with branch cut along the negative imaginary axis and  $\omega^-(z) = z^{(1/2)}$  with branch cut along the positive imaginary axis. Finally, one may construct  $X_3^\pm(z)$  by

$$X_3^\pm(z) = \exp(r^\pm(z)) \text{ where } r^\pm(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log(G_3(r))}{r-z} dr. \quad (2.18)$$

To find a closed form expression for  $X_s^{\pm}(z)$ , it is necessary to determine the mapping properties of  $G_s(z)$  for  $z$  in the half-plane  $\text{Im}(z) \leq 0$ . If one first considers  $\tilde{\mu}(s+ivz) = \tilde{\mu}(s-q+ivp)$  on the horizontal lines  $z=p+iq$ ,  $q \leq 0$ ,  $p \in (-\infty, \infty)$ , then it follows easily from the stated assumptions on  $\mu(t)$  that

- (i)  $\tilde{\mu}(0) = \mu(\infty) < \tilde{\mu}(s-q) \leq \text{Re } \tilde{\mu}(s-q+ivp) \leq \mu(0) = \tilde{\mu}(s-q+i\infty)$ ;
- (ii)  $\text{Im } \tilde{\mu}(s-q+ivp) = -\text{Im } \tilde{\mu}(s-q-ivp)$ ;
- (iii)  $\arg \tilde{\mu}(s-q+ivp) \begin{cases} > 0 & p > 0 \\ < 0 & p < 0 \end{cases}$ ;
- (iv)  $\lim_{q \rightarrow -\infty} \tilde{\mu}(s-q+ivp) = \mu(0)$  with  $\tilde{\mu}(s-q)$  converging monotonically to  $\mu(0)$ .

Therefore  $\tilde{\mu}(s+ivz)$  maps these lines to the curves shown in fig 2.2a.

The linear fractional transformation  $S(z) = \frac{z}{z - i(s/v)}$  maps the lines  $z=p+iq$ ,  $q \leq 0$ ,  $p \in (-\infty, \infty)$  to the circles shown in fig 2.2b and  $S(p+iq)$  exhibits the following properties:

- (v)  $0 = S(0) \leq S(0+iq) \leq \text{Re } S(p+iq) \leq S(\pm\infty+iq) = 1$ ;
- (vi)  $\text{Im } S(p+iq) = -\text{Im } S(-p+iq)$ ;
- (vii)  $\arg S(p+iq) \begin{cases} > 0 & p > 0 \\ < 0 & p < 0 \end{cases}$ ;
- (viii)  $\lim_{q \rightarrow -\infty} S(p+iq) = 1$  with  $S(iq)$  converging monotonically to 1.

If the elastic shear wave speeds corresponding to infinite and zero time are defined by  $c_*^2 = \mu(\infty)/\rho$  and  $c^2 = \mu(0)/\rho$  then from (i)-(viii), it follows that  $[G_s(z)]^2$  has

the following properties:

$$(ix) \quad \text{Im } G_s^2(p+iq) = -\text{Im } G_s^2(-p+iq);$$

$$(x) \quad \arg G_s^2(p+iq) \begin{cases} > 0 & p > 0 \\ < 0 & p < 0 \end{cases};$$

$$(xi) \quad \lim_{q \rightarrow -\infty} G_s^2(p+iq) = 1 - (v/c)^2;$$

$$(xii) \quad -(v/c_*)^2 = G_s^2(0) \leq G_s^2(iq) \leq G_s^2(-i\infty) = 1 - (v/c)^2 \text{ where } G_s^2(iq) \text{ is monotonically increasing to } 1 - (v/c)^2 \text{ as } q \rightarrow -\infty.$$

Thus it can be seen that  $G_s^2(z)$  maps the horizontal lines  $z=p+iq$ ,  $q \leq 0$ ,  $p \in (-\infty, \infty)$  to the curves shown in fig 2.2c.

Furthermore, it can be seen from (xii) that  $G_s(z)^2$  has a unique root  $z_* = iq_*$  in the half-plane  $\text{Im}(z) \leq 0$  for any positive real value of  $s$ .

If the branch cut for the square root defining  $G_s(z)$  is chosen along the negative real axis then  $G_s(z)$  is analytic for  $\text{Im}(z) < 0$  except for the branch cut on a segment of the negative imaginary axis across which it has a jump

$$\text{discontinuity given by } G_s(\pm 0+iq) = \pm i \left[ \frac{-q}{(s/v) - q} \right] \text{ for } q_* < q \leq 0.$$

It follows that  $\log G_s(z)$  is analytic for  $\text{Im}(z) < 0$  away from this line segment and has the jump discontinuity

$$\log(G_s(+0+iq)) - \log(G_s(-0+iq)) = i\pi \text{ across } z=iq, \quad q_* < q \leq 0. \text{ One}$$

$$\text{can then evaluate } r^\pm(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log(G_s(r))}{r-z} dr \text{ as in}$$

Walton(1982) and conclude that

$$r^\pm(z) = (1/2) \left[ \log(q_* i - z) - \log(-z) + \log(G_s(z)) \right. \\ \left. + \begin{cases} 0 & \text{for } \text{Im}(z) > 0 \\ -\log(G_s(z)) & \text{for } \text{Im}(z) < 0 \end{cases} \right].$$

Therefore (2.18) reduces to

$$X_+^+(z) = i\omega^+(z - iq_*)[G_+(\infty)]^{(1/2)} / \omega^+(z) \quad (2.19)$$

$$X_+^-(z) = i\omega^-(z - iq_*)[G_+(\infty)]^{(1/2)} / \omega^-(z) G_+(z).$$

From equations (2.16)-(2.19) it then follows that

$$X^+(z) = i\omega^+(z - iq_*)[G_+(\infty)]^{(1/2)} \quad (2.20)$$

$$X^-(z) = -i\omega^-(z - iq_*)[G_+(\infty)]^{(1/2)} / [\tilde{\mu}(s + ivz)(z - i(s/v))G_+(z)].$$

Finally, for a specific load  $\sigma^-(x) = f(x)$  one can

determine  $F^+(z) = \bar{\sigma}_2^+(z, 0, s)$ ,  $F^-(z) = \bar{w}^-(z, 0, s)$  from equations (2.13) and (2.20). The Laplace transform of the stress intensity factor (SIF)  $\bar{K}(s)$  can now be calculated as in Walton(1982). In particular, it is straight forward to show that

$$\bar{\sigma}(x) \sim \frac{\bar{K}(s)}{\sqrt{\pi}} x^{(-1/2)} \text{ as } x \rightarrow 0^+ \quad (2.21)$$

$$\text{where } \bar{K}(s) = [G_+(\infty)]^{(1/2)} \frac{e^{(-1/4)\pi i}}{2\pi} \int_{-\infty}^{\infty} \frac{\bar{f}(\tau)}{X^+(\tau)} d\tau.$$

Again, as in Walton(1982), (2.21) may be simplified to

$$\bar{K}(s) = \frac{-1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{f(x)}{s} |x|^{(-1/2)} e^{-xq_*} dx. \quad (2.22)$$

in which it should be emphasized that  $q_*$  is a function of  $s$  determined implicitly by  $G_+(iq_*) = 0$ .

### 3. Calculation of the Energy Release Rate.

The Energy Release Rate (ERR) will now be calculated based upon the assumption that a Barenblatt type failure zone exists at the crack-tip. Specifically, it is assumed that two loads are acting on the crack-faces: the applied (external) tractions denoted  $\sigma_e^-(x)$  and the cohesive (failure) stresses  $\sigma_f^-(x)$  acting in a failure zone of length  $a_f$  immediately behind the crack-tip. The only assumptions about  $\sigma_f^-(x)$  are that  $a_f$  is small relative to some length scale  $a_e$  associated with  $\sigma_e^-(x)$  and that  $K_e + K_f = 0$  where  $K_e$  and  $K_f$  are the SIF's corresponding to  $\sigma_e^-$  and  $\sigma_f^-$ , respectively. Hence the effect of the failure zone is to cancel the singular stresses ahead of the crack-tip and thereby produce a cusp shaped crack profile behind the tip.

The ERR,  $G(t)$  (defined to be the energy flux into the crack tip per unit crack advance) is given by

$$G(t) = \frac{1}{v} \int_{vt-a_f}^{vt} \sigma_f^-(x, -vt) \dot{u}_2(x, -vt, 0, t) dx,$$

which in the moving coordinates becomes

$$G(t) = \frac{1}{v} \int_{-a_f}^0 \sigma_f^-(x) \left[ \frac{\partial}{\partial t} - v \frac{\partial}{\partial x} \right] w(x, 0, t) dx. \quad (3.1)$$

It is impractical to calculate (3.1) directly since the expression  $\left[ \frac{\partial}{\partial t} - v \frac{\partial}{\partial x} \right] w(x, 0, t)$  has a very complicated dependence on  $\sigma_e^-(x)$  and  $\sigma_f^-(x)$  through the inverse Laplace

transform of (2.13) and (2.20). A computationally convenient expression will be derived for the Laplace transform of  $G(t)$ ,  $\bar{G}(s)$ , for the special loadings  $\sigma_e$  and  $\sigma_f$  used in Walton(1987a). Specifically, it is assumed that

$$\sigma_e^-(x) = L_e \exp(x/a_e) H(-x) \text{ and } \sigma_f^-(x) = -L_f \exp(x/a_f) H(-x) \quad (3.2)$$

where  $a_f/a_e \ll 1$ . For  $a_f/a_e$  small enough, the fact that  $\sigma_f^-(x)$  does not have support in some small, compact interval behind the crack-tip will have a negligible effect on the results. The assumptions (3.2) clearly incorporate the salient features of the Barenblatt model, namely an applied load  $\sigma_e^-$  paired with a cohesive stress  $\sigma_f^-$ , each with an associated length scale  $a_e$  and  $a_f$  respectively, such that  $\sigma_f^-$  cancels the singular stress produced by  $\sigma_e^-$  and  $a_f/a_e \ll 1$ . It should be noted that in this case (3.1) should be replaced by

$$G(t) = \frac{1}{v} \int_{-\infty}^0 \sigma_f^-(x) \left[ \frac{\partial}{\partial t} - v \frac{\partial}{\partial x} \right] w(x, 0, t) dx. \quad (3.3)$$

As shown in Walton (1987a), it is straight forward to extend the analysis to treat more general loads in the form

$$\sigma^-(x) = L \int_0^{\infty} e^{tx/a} dh(t) \text{ where } h(t) \text{ is any signed measure for}$$

which the integral makes sense. However, for the sake of brevity that development is not included here.

If one applies Parseval's formula and then takes the Laplace transform of the resulting expression, the Laplace transform of the ERR is found to be given by

$$\bar{G}(s) = \frac{1}{v} \int_{-\infty}^{\infty} \sigma_f^-(p) \left[ s + ipv \right] \bar{w}^-(p, 0, s) dp \quad (3.4)$$

where  $\sigma_f^-(p)$  denotes the inverse Fourier transform of  $\sigma_f^-(x)$ .

For the particular choice of  $\sigma_f^-(x)$  given in (3.2),

$$\sigma_f^-(p) = \frac{L_f}{2\pi i(p + (1/a_f))}. \quad \text{It is easily seen that } \sigma_f^-(p) \text{ has a}$$

meromorphic extension  $\sigma_f^-(z)$  with a simple pole at  $z = -1/a_f$ .

Furthermore, since  $\bar{w}^-(z, 0, s) = F^-(z)$  is analytic for  $\text{Im}(z) < 0$ ,

(3.4) can be evaluated by residues and whence it follows that

$$\bar{G}(s) = \frac{-L_f}{v} (s + (v/a_f)) F^-(-1/a_f). \quad (3.5)$$

It remains to evaluate  $F^-(-1/a_f)$ . To this end, one begins by first noting that (2.11) can be rewritten as

$$F^-(p) = \bar{\sigma}^-(p) / T(p) \quad (3.6)$$

$$\text{where } \bar{\sigma}^-(p) = \bar{\sigma}^-(p) + \bar{\sigma}^+(p). \quad (3.7)$$

From the Barenblatt model  $\sigma^-(x) = \sigma_e^-(x) + \sigma_f^-(x)$  and consequently

$$\bar{\sigma}^-(p) = \bar{\sigma}_e^-(p) + \bar{\sigma}_f^-(p). \quad (3.8)$$

Also  $\bar{\sigma}^+(p)$  in (3.7) can be determined by application of the Plemelj formula to (2.13) thereby obtaining

$$\bar{\sigma}^+(p) = F^+(p) = -\frac{1}{2} \bar{f}(p) + \frac{1}{2\pi i} X^+(p) \int_{-\infty}^{\infty} \frac{-\bar{f}(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - p}. \quad (3.9)$$

This can be rewritten using

$$\bar{\sigma}^+(p) = \bar{\sigma}_e^+(p) + \bar{\sigma}_f^+(p) \quad (3.10)$$



wherein  $\bar{\sigma}_e^+(p)$  and  $\bar{\sigma}_f^+(p)$  are given by

$$\bar{\sigma}_\alpha^+(p) = -\frac{1}{2}\bar{\sigma}_\alpha^-(p) + \frac{1}{2\pi i} X^+(p) \int_{-\infty}^{\infty} \frac{-\bar{\sigma}_\alpha^-(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - p}. \quad (3.11)$$

From (3.2) it can be seen that  $\bar{\sigma}_e^-(p) = \frac{L_e}{is(p - (i/a_e))}$  and

therefore  $\bar{\sigma}_e^-(z)$  is meromorphic with pole  $z = i/a_e$ . Since  $X^+(z)$  is analytic for  $\text{Im}(z) > 0$ , (3.11) can be calculated by residues with the result

$$\bar{\sigma}_e^+(p) = -\bar{\sigma}_e^-(p) \left[ 1 - \frac{X^+(p)}{X^+(i/a_e)} \right]. \quad (3.12)$$

Similarly,

$$\bar{\sigma}_f^+(p) = -\bar{\sigma}_f^-(p) \left[ 1 - \frac{X^+(p)}{X^+(i/a_f)} \right] \quad (3.13)$$

which combined with (3.10) and (3.12) yields

$$\bar{\sigma}(p) = \frac{X^+(p)}{is} \left[ \frac{L_e}{(p - (i/a_e))X^+(i/a_e)} - \frac{L_f}{(p - (i/a_f))X^+(i/a_f)} \right]. \quad (3.14)$$

Equation (3.14) can now be simplified by the Barenblatt hypothesis  $K_e + K_f = 0$ . For the special loadings given by (3.2), equation (2.21) yields

$$\bar{K}_e(s) = [G, (\infty)]^{(1/2)} e^{(-1/4)\pi i} \frac{L_e}{sX^+(i/a_e)} \quad \text{and}$$

$$\bar{K}_f(s) = [G, (\infty)]^{(1/2)} e^{(-1/4)\pi i} \frac{-L_f}{sX^+(i/a_f)}, \quad \text{which under the}$$

Barenblatt hypothesis results in the identity

$$\frac{L_e}{X^+(i/a_e)} = \frac{L_f}{X^+(i/a_f)}. \quad (3.15)$$

Equations (3.14), (3.15), (3.6) and the observation

that  $T(p) = X^+(p)/X^-(p)$  then yield that

$$F^-(-i/a_f) = \frac{a_f L_e}{2s} \frac{(a_e - a_f)}{(a_e + a_f)} \frac{X^-(-i/a_f)}{X^+(i/a_e)}. \quad (3.16)$$

One may now substitute (3.15) and (3.16) into (3.5) thereby obtaining

$$\bar{G}(s) = \frac{L_e^2}{2vs} (s + (v/a_f)) \frac{a_f(a_f - a_e)}{(a_e + a_f)} \frac{X^-(-i/a_f) X^+(i/a_f)}{[X^+(i/a_e)]^2}, \quad (3.17)$$

which when combined with the result (2.20) derived previously for  $X^\pm(z)$  produces the desired expression for the Laplace transform of the ERR

$$\bar{G}(s) = \frac{L_e^2 a_e (a_e - a_f)}{2s (a_e + a_f)} \frac{|1 - (a_f q_*)^2|^{(1/2)}}{[1 - a_e q_*]} \frac{1}{\tilde{\mu}(s + v/a_f)} \left| \frac{1}{(1 + s a_f/v)^2} - \frac{\rho v^2}{\tilde{\mu}(s + v/a_f)} \right|^{-(1/2)}. \quad (3.18)$$

In order to present the results in a nondimensional form, it is necessary to introduce certain parameters.

First, a nondimensional shear modulus is defined by

$$\mu(t) = \mu_\infty m(t/\tau) \text{ where } \mu_\infty = \lim_{t \rightarrow \infty} \mu(t) \text{ and thus } \lim_{t \rightarrow \infty} m(t) = 1. \text{ Also,}$$

the nondimensional parameters  $\gamma$ ,  $\epsilon$ ,  $\alpha$ , and  $\beta$  are defined by

$$\gamma = v/c_*, \quad \epsilon = a_f/a_e, \quad \alpha = c_* \tau/a_e, \quad \text{and } \beta = -a_e q_*, \text{ where } 0 < \gamma < c/c_*, \quad \epsilon \ll 1, \quad \alpha > 0, \text{ and } \beta > 0.$$

The Laplace transform of the ERR can now be rewritten in nondimensional form as

$$s\bar{G}(s) = \frac{L e^{2a} e}{2\gamma\mu_{\infty}} \frac{(1-\epsilon)}{(1+\epsilon)} \frac{|1-(\epsilon\beta)^2|^{(1/2)}}{(1+\beta)} \frac{1}{\tilde{m}(\tau s + \alpha\gamma/\epsilon)} \left| \frac{(\alpha/\epsilon)^2}{(\tau s + \alpha\gamma/\epsilon)^2} - \frac{1}{\tilde{m}(\tau s + \alpha\gamma/\epsilon)} \right|^{-(1/2)}. \quad (3.19)$$

It is the study of this basic expression for  $\bar{G}(s)$  found in (3.19) that leads to the special cases and asymptotic approximations found in the next section.

#### 4. Short-time and Long-time Asymptotic Solutions.

Asymptotic expansions for the ERR,  $G(t)$ , as  $t \rightarrow 0$  and as  $t \rightarrow \infty$  can now be constructed from expression (3.19) for the Laplace transform of the ERR. For the short time approximation, it is necessary to determine the asymptotic behavior of  $\bar{G}(s)$  as  $s \rightarrow \infty$ . To this end, it is required to study the behavior of  $\tilde{m}(rs + \alpha\gamma/\epsilon)$  and  $\beta = -a_e q_*$  as  $s \rightarrow \infty$ . From the definition of the Carson transform  $\tilde{m}(s)$ , it is easily seen that

$$\tilde{m}(\infty) = m(0) = \frac{\mu(0)}{\mu_\infty} = \left[ \frac{c}{c_*} \right]^2. \quad (4.1)$$

Furthermore, since  $iq_*$  is the unique root of  $G_2(z) = 0$  in the lower half-plane  $\text{Im}(z) < 0$ ,  $\beta = \beta(s, \gamma)$  satisfies the equation

$$\tilde{m}(rs + \alpha\gamma\beta) = \gamma^2 (1 + (rs/\alpha\gamma\beta))^2. \quad (4.2)$$

In the appendix, it is shown from (4.2) that

$$\beta = \frac{rs}{\alpha[(c/c_*) - \gamma]} + o(s) \text{ as } s \rightarrow \infty. \quad (4.3)$$

Therefore if  $\bar{G}(s)$  is written as an asymptotic series in powers of  $\frac{1}{s}$ , it is found that as  $s \rightarrow \infty$

$$\bar{G}(s) = \frac{L e^{2a_e} \epsilon (1-\epsilon)}{2\gamma\mu_\infty} \frac{c_*}{c} \left[ \frac{1}{s} - \frac{1}{\gamma} \left[ \alpha[(c/c_*) - \gamma] + \frac{m'(0)}{2m(0)} \right] \frac{1}{s^2} + o(s^{-2}) \right]. \quad (4.4)$$

Hence, if one assumes that  $G(t)$  has a Maclaurin expansion in a neighborhood of  $t=0$  then from (4.4) and standard asymptotic results for the Laplace transform,

$$G(t) = \frac{L e^{2a_e} \epsilon (1-\epsilon)}{2\mu_\infty} \frac{c_*}{\gamma} \frac{1}{c} \left[ 1 - \left[ \alpha[(c/c_*) - \gamma] + \frac{m'(0)}{2m(0)} \right] \frac{t}{\gamma} + o(t) \right] \text{ as } t \rightarrow 0. \quad (4.5)$$

At this point, it is worth noting that by allowing  $\epsilon \rightarrow 0$  in (3.19), one obtains the ERR for the singular stress field, i.e. the energy flux into the crack-tip per unit crack advance in the absence of a failure zone. (If one specializes further to steady-state conditions, one obtains Atkinson's expression for the ERR based on a local work argument at the crack-tip.) One then finds that in this case

$$s\bar{G}(s) = \frac{L e^2 a_e}{2\mu_\infty} \frac{1}{(1+\beta)} \frac{1}{\bar{m}(0)} \left| 1 - \frac{\gamma^2}{\bar{m}(0)} \right|^{-(1/2)} \\ = \frac{1}{2\mu_0} [1 - (v/c)^2]^{(-1/2)} [s\bar{K}(s)]^2 \text{ where } \bar{K}(s) \text{ is the Laplace} \\ \text{transform of the SIF. This can be inverted and produces} \\ G(t) = \frac{1}{2\mu_0} [1 - (v/c)^2]^{(-1/2)} [K(t) * K'(t)] \quad (4.6)$$

since  $K(0) = K(0+) = 0$ . Therefore with no Barenblatt zone

$$G(t) = \frac{L e^2 a_e}{2\mu_0} [1 - (v/c)^2]^{(-1/2)} \left[ [1 - (v/c)] \frac{t}{\tau_1} + o(t) \right] \text{ as } t \rightarrow 0 \quad (4.7)$$

where  $\tau_1 \equiv a_e/c$ .

To determine long-time asymptotic solutions as  $t \rightarrow \infty$ , it is necessary to first find  $\lim_{t \rightarrow \infty} G(t) = \lim_{s \rightarrow 0} s\bar{G}(s)$ . The expression  $\tilde{m}(\gamma s + \alpha \gamma / \epsilon)$  has the limit  $\tilde{m}(\alpha \gamma / \epsilon)$  as  $s \rightarrow 0$  and  $\beta$ , as shown in the appendix, has different asymptotic limits as  $s \rightarrow 0$  depending on whether (i)  $0 < v < c_*$ , (ii)  $v = c_*$ , or (iii)  $c_* < v < c$ , namely

$$\beta = \frac{\gamma s}{\alpha[1-\gamma]} + o(s) \quad \text{for } 0 < v < c_* \quad (4.8)$$

$$\beta = \frac{\sqrt{2\gamma}}{\alpha\gamma} \left[ -\int_0^\infty r m'(r) dr \right]^{(-1/2)} s^{1/2} + o(s^{1/2}) \quad \text{for } v = c_*,$$

$$\beta = \beta_0 + o(1) \quad \text{for } c_* < v < c,$$

where  $\beta_0$  is defined implicitly through the equation

$$\gamma^2 - \left[ \frac{c}{c_*} \right]^2 = \int_0^\infty e^{-\alpha\gamma\beta_0 r} m'(r) dr. \quad (4.9)$$

To avoid separate cases in displaying subsequent formulas it is convenient to define  $\beta_0 = 0$  for  $0 < v \leq c_*$ .

Again, from standard asymptotic arguments for the Laplace transform

$$\lim_{t \rightarrow \infty} G(t) = \frac{L e^2 a_e}{2\mu_\infty} \frac{(1-\epsilon)}{(1+\epsilon)} \frac{|1-(\epsilon\beta_0)^2|^{(1/2)}}{(1+\beta_0)} \frac{1}{\tilde{m}(\alpha\gamma/\epsilon)} \left| 1 - \frac{\gamma^2}{\tilde{m}(\alpha\gamma/\epsilon)} \right|^{-(1/2)} \quad (4.10)$$

which is precisely the steady-state solution found in Walton(1987a).

Several comments on these asymptotic results are in order. First, it should be observed that whether or not the model assumes a failure zone dramatically affects both qualitatively and quantitatively the behavior of the ERR as a function of time, crack speed, and material properties. For example, from (3.19) it is easily shown that for any  $t$ ,

$$\text{if } \epsilon > 0, \lim_{t \rightarrow \infty} G(t) = \infty \text{ whereas if } \epsilon = 0, \lim_{t \rightarrow \infty} G(t) = \frac{L e^2 a_e}{2\mu_0} D(t), \text{ where}$$

$$s\bar{D}(s) = \frac{1}{1+\beta(s,0)} \text{ and from (4.2) } \beta(s,0) = \frac{\gamma s}{\alpha} [\tilde{m}(\gamma s)]^{(-1/2)}.$$

However, in the steady-state limit (4.10),  $G$  approaches a finite limit as the crack speed vanishes, both with and

without a failure zone. Specifically, it is easily seen that for the steady-state limit

$$\lim_{v \rightarrow 0} G(\infty) = \frac{L e^{2a_e}}{2} \frac{1-\epsilon}{1+\epsilon} \frac{1}{\mu_\infty} \quad \text{for } \epsilon > 0 \quad (4.11)$$

$$= \frac{L e^{2a_e}}{2} \frac{1}{\mu_0} \quad \text{for } \epsilon = 0.$$

Thus  $G$  becomes infinite as  $v \rightarrow 0$  except for  $\epsilon = 0$  (no failure zone) or under steady-state conditions.

The reason for this behavior is found through consideration of the crack face particle velocity  $\dot{u}_s(x, 0, t) = \frac{\partial w}{\partial t}(x, 0, t) - v \frac{\partial w}{\partial x}(x, 0, t)$ . A consequence of the assumption that there is an initial jump discontinuity in the applied crack face tractions is that  $\frac{\partial w}{\partial t}(x, 0, t)$  does not vanish as  $v \rightarrow 0$ . Thus, from (3.3) it follows that  $\lim_{v \rightarrow 0} G(t) = \infty$ .

In contrast, when  $\epsilon = 0$ , one sees from (4.6) that  $G$  is merely a product of  $K(t) * K'(t)$ , which depends only on the SIF and a simple function of crack speed and glassy material properties that is independent of the crack face particle velocity and that remains bounded as  $v \rightarrow 0$ . Moreover, in steady-state,  $G$  is given by  $G = - \int_{-a_f}^0 \sigma_f^-(x) \frac{\partial w}{\partial x}(x, 0) dx$  and thus remains bounded as  $v \rightarrow 0$ .

Other differences between the  $\epsilon = 0$  and  $\epsilon > 0$  cases are evident in the short time behavior of  $G(t)$ . In particular, from (4.5) it is easily seen that when  $\epsilon > 0$ ,

$$\lim_{t \rightarrow 0} G(t) = \frac{L e^2 a_e}{2\mu_\infty} \frac{1-\epsilon}{1+\epsilon} \frac{\epsilon}{\gamma} \frac{c_*}{c} > 0 \quad (4.12)$$

whereas, for  $\epsilon=0$ , it follows from (4.7) that

$$\lim_{t \rightarrow 0} G(t) = 0.$$

Additionally from (4.5), when  $\epsilon > 0$ ,  $G'(0)$  is given by

$$G'(0) = -\frac{L e^2 a_e}{2\mu_\infty} \frac{(1-\epsilon)}{(1+\epsilon)} \frac{\epsilon}{\gamma} \left[ \alpha [1-(v/c)] + \frac{m'(0)}{2m(0)} (c_*/c) \right] \frac{1}{\gamma}. \quad (4.13)$$

From the fact that  $m'(0) < 0$  and using the identities

$\frac{m'(0)}{m(0)} = \frac{\mu'(0)}{\mu(0)}$  and  $\alpha = c_* \tau / a_e$ , it follows that the sign of  $G'(0)$  depends upon the crack speed and material properties through the relation

$$G'(0) > 0 \left[ G'(0) < 0 \right] \text{ if and only if} \quad (4.14)$$

$$\frac{|\mu'(0)|}{2\mu(0)} (a_e/c) > [1-(v/c)] \left[ \frac{|\mu'(0)|}{2\mu(0)} (a_e/c) < [1-(v/c)] \right].$$

In particular for fast enough crack speeds,  $G'(0)$  will be positive, i.e.  $G(t)$  will initially increase with time.

However, for any given crack speed if the combination

$\frac{|\mu'(0)|}{2\mu(0)} (a_e/c)$  is small enough, then  $G(t)$  will initially decrease with time.

Attention will now be directed toward describing the manner in which  $G(t)$  converges to the steady state limit as  $t$  approaches infinity. It is useful for comparison purposes to consider the question first for elastic material. An expression for  $s\bar{G}(s)$  for elastic material emerges by letting  $\tau \rightarrow \infty$ ,  $\mu_\infty = \mu_0 = \mu$  and  $\tilde{m}(s) \equiv 1$  in (3.19). Recalling the identity  $\alpha = c_* \tau / a_e$  and noting that  $c = c_*$ , one sees easily that



$$s\bar{G}(s) = \frac{L_e^2 a_e}{2\mu\gamma} \frac{(1-\epsilon)}{(1+\epsilon)} \frac{|1-(\epsilon\beta)^2|^{(1/2)}}{(1+\beta)} \left| \frac{1}{(\gamma+\epsilon\tau_1 s)^2} - 1 \right|^{-(1/2)} \quad (4.15)$$

where  $\tau_1 = a_e/c$ . Moreover, from (3.20) it is straight forward to deduce that for elastic material

$$\beta = s\tau_1 / (1-\gamma). \quad (4.16)$$

Substitution of (4.16) into (4.15) and routine algebraic manipulation yields

$$s\bar{G}(s) = \frac{L_e^2 a_e}{2\mu} \frac{(1-\epsilon)}{(1+\epsilon)} (1-\gamma^2)^{(-1/2)} (1 + \frac{s\tau_1 \epsilon}{1-\gamma})^{(1/2)} (1 + s\tau_1 \epsilon / \gamma)^{-1} (1 + \frac{s\tau_1}{1-\gamma})^{-1} (1 + \frac{s\tau_1 \epsilon}{1+\gamma})^{(-1/2)}. \quad (4.17)$$

It is instructive to consider first the elastic limit with no failure zone. Letting  $\epsilon \rightarrow 0$  in (4.17), one then has

$$s\bar{G}(s) = \frac{L_e^2 a_e}{2\mu} (1-\gamma^2)^{(-1/2)} (1 + \frac{s\tau_1}{1-\gamma})^{-1} \text{ from which it follows that } G(t) = \frac{L_e^2 a_e}{2\mu} (1-\gamma^2)^{(-1/2)} (1 - e^{-a_1 t}) \text{ where } a_1 = (1-\gamma)/\tau_1 = (c-v)/a_e.$$

Thus,  $G(t)$  converges to its steady state limit with an exponential decay rate that decreases monotonically with increasing crack speed up to the shear wave speed.

A somewhat more complicated expression for  $G(t)$  results for  $\epsilon > 0$ . Calculating  $G(t)$  by Laplace inversion of (4.17) may be conveniently done by defining  $a_1 = (1-\gamma)/\tau_1$  (as above),  $a_2 = \gamma/(\tau_1 \epsilon)$ ,  $a_3 = a_1/\epsilon$  and  $a_4 = (1+\gamma)/(\tau_1 \epsilon)$ , and then considering the Laplace inversion of

$$H(s) = \frac{1}{s} \frac{(1+s/a_2)}{(1+s/a_1)} (1+s/a_3)^{(1/2)} (1+s/a_4)^{(-1/2)}. \text{ It is}$$

convenient to factor  $H(s)$  as the product  $H(s) = H_1(s)H_2(s)$

$$\text{with } H_1(s) = \frac{(1+s/a_2)(1+s/a_3)}{s(1+s/a_1)} \text{ and}$$

$H_2(s) = \sqrt{a_1 a_4} \left[ [s + (a_1 + a_4)/2]^2 - [(a_1 - a_4)/2]^2 \right]^{-1/2}$ . Then  $H(s)$  can be inverted to  $h(t) = h_1 * h_2$  where  $h_1(t) = b_1 \delta(t) + 1 + b_2 e^{-a_1 t}$  and  $h_2(t) = \frac{(1-\gamma^2)^{1/2}}{\tau_1 \epsilon} e^{-(1/\epsilon)(t/\tau_1)} I_0\left(\frac{\gamma t}{\epsilon \tau_1}\right)$  where  $b_1 = \frac{\tau_1 \epsilon^2}{\gamma}$ ,  $b_2 = -1 + (\epsilon/\gamma)(1 - \epsilon(1-\gamma))$ ,  $\delta(t)$  is the Dirac point measure, and  $I_0(t)$  is the modified Bessel function of the first kind of

order zero defined by  $I_0(t) = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}t)^{2r}}{(r!)^2}$ . Therefore the ERR

may be written as

$$G(t) = \frac{L^2 a_1 e^{(1-\epsilon)} (1-\gamma^2)^{-1/2}}{2\mu (1+\epsilon)} \left[ b_1 h_2(t) + \int_0^t h_2(t) dt + b_2 e^{-a_1 t} * h_2(t) \right] \quad (4.18)$$

From the fact that  $0 < a_1 < a_3 < a_4$ , an easy argument shows that  $\frac{d}{dt}h(t)$  decays like  $e^{-a_1 t}$  as  $t \rightarrow \infty$ . Thus  $G(t)$  converges to its steady-state limit with the same exponential rate as in the  $\epsilon=0$  case.

It is easily seen that for a viscoelastic material the situation is considerably more complicated due to the combined influence of material inertia and viscoelastic stress relaxation upon convergence to steady state. A general property of the Laplace transform  $F(s)$  of a function  $f(t)$  is that  $f(t)$  decays exponentially in time, say  $f(t) \sim e^{-at}$  with  $a > 0$ , if and only if  $F(s)$  is analytic in the halfplane  $\text{Re}(s) > -a$ . The expression (3.19) for  $s\bar{G}(s)$  is valid for  $s$  real. Its analytic extension for complex  $s$  is given by

$$s\bar{G}(s) = \frac{L e^{2a}}{2\gamma\mu_{\infty}} \frac{(1-\epsilon)}{(1+\epsilon)} (1+\beta)^{-1} [\tilde{m}(\tau s + \alpha\gamma/\epsilon)]^{-1} \left[ [1-(\epsilon\beta)^2] / \left[ \frac{1}{(\gamma + \epsilon\tau s/\alpha)^2} - \frac{1}{\tilde{m}(\tau s + \alpha\gamma/\epsilon)} \right] \right]^{(1/2)}. \quad (4.19)$$

Determining the largest value of  $a$ ,  $a_{\max}$ , for which  $s\bar{G}(s)$  is analytic in the halfplane  $\text{Re}(s) > -a$  is a difficult task that clearly depends upon the particular details of the transform  $\tilde{m}(\tau s + \alpha\gamma/\epsilon)$ . From the previously stated properties assumed for  $m(t)$  it follows that  $[\tilde{m}(\tau s + \alpha\gamma/\epsilon)]^{-1}$  is analytic for  $\text{Re}(s) > -\frac{\alpha\gamma}{\tau\epsilon}$ . If  $m(t)$  is a powerlaw in  $t$ , then  $\tilde{m}(\tau s + \alpha\gamma/\epsilon)$  has  $s = -\frac{\alpha\gamma}{\tau\epsilon}$  as a branch point and  $a_{\max}$  can be no larger than that. As a second example, for a standard linear solid with  $m(t) = 1 + \eta e^{-t}$ ,  $[\tilde{m}(\tau s + \alpha\gamma/\epsilon)]^{-1}$  is analytic for  $\text{Re}(s) > -[(\eta + 1)^{-1} + \alpha\gamma/\epsilon]/\tau$ .

The heart of the matter lies in determining the analyticity properties of  $\beta(s, \gamma)$ , which is defined implicitly through equation (4.2). Whether or not  $\beta(s, \gamma)$  is analytic in some halfplane  $\text{Re}(s) > -a$ ,  $a > 0$ , depends upon the particular way  $m(t)$  decays to its equilibrium value as  $t \rightarrow \infty$ . This necessitates a case by case analysis for different forms of  $m(t)$ . We content ourselves here with illustrating the differences that exist between materials with exponentially decaying modulus, such as a standard linear solid, and those for which  $m(t)$  decays as a power of  $t$  to its equilibrium value, such as a simple powerlaw material with  $m(t) = 1 + \eta(1+t)^{-n}$ ,  $n > 0$ .

An important observation to be made for powerlaw material is that  $G(t)$  cannot have exponential decay to its steady-state limit when  $\gamma < 1$ , i.e.  $v < c_*$ . The reason for this is that  $\beta(s, \gamma)$  is not analytic at  $s=0$  as can be seen from the following argument. It was remarked earlier that  $\beta(0, \gamma) = 0$  whenever  $\gamma < 1$ , in particular  $\lim_{s \rightarrow 0+} \beta(s, \gamma)/s = \frac{\gamma}{\alpha(1-\gamma)}$ . Moreover, for powerlaw material  $\tilde{m}(s)$  has  $s=0$  as a branch point. If  $\beta(s, \gamma)$  were analytic at  $s=0$  then the right hand side of equation (4.2) would also be analytic there. However, the left hand side of (4.2) has  $s=0$  as a branch point. This contradiction proves the claim.

On the other hand, for  $1 < \gamma < m(0)^{(1/2)}$  (i.e.  $c_* < v < c$ ), since  $\beta(0, \gamma) > 0$ , equation (4.2) defines  $\beta(s, \gamma)$  as an analytic function in a neighborhood of  $s=0$ . In particular,  $\beta(s, \gamma)$  is analytic in a halfplane  $\text{Re}(s) > -a$ ,  $a > 0$ . However, finding the largest such  $a$  is difficult. For elastic material, the rate of exponential convergence of  $G(t)$  to its steady-state limit corresponds to the negative real value of  $s$  for which  $\beta(s, \gamma) = -1$ . For viscoelastic materials, again a case by case analysis will be required to determine precisely where the singularity of  $s\bar{G}(s)$  with largest real part will occur.

The remaining case  $\gamma = 1$  (i.e.  $v = c_*$ ) is easily handled. Indeed, for a general material, not just powerlaw or a standard linear solid, equation (4.2) admits no solutions for  $s < 0$ . Thus  $\beta(s, 1)$  is not analytic in any halfplane

$\operatorname{Re}(s) > -a$ ,  $a > 0$  and  $G(t)$  cannot converge exponentially to its steady-state limit. Furthermore, it is not difficult to see from equation (4.2) that the left endpoint of the largest  $s$ -interval containing zero for which an admissible solution  $\rho(s, \gamma)$  exists converges to zero as  $\gamma$  approaches 1 from above.

These observations made for powerlaw material whenever  $1 \leq \gamma < m(0)^{(1/2)}$  ( $c_* < v < c$ ) are equally valid for a standard linear solid. A departure in behavior occurs for  $0 < \gamma < 1$ . If  $m(t) = 1 + \eta e^{-t}$ , then  $\tilde{m}(s) = \frac{1 + (1 + \eta)s}{1 + s}$  and (4.2) can be rewritten as

$$\frac{1 + (1 + \eta)z}{1 + z} = \gamma^2 \left[ \frac{z}{z - \tau s} \right]^2, \quad (4.20)$$

where  $z = \tau s + \alpha \gamma \rho(s, \gamma)$ . When  $s > 0$ , a root  $z$  must be sought that is greater than  $\tau s$ , whereas  $z < \tau s$  is required when  $s < 0$  is suitably near zero. An examination of the graphs of the functions on either side of equation (4.20) quickly reveals that admissible solutions exist for any  $s > 0$  and  $s < 0$  suitably near zero. Moreover, no solution exists for  $\gamma = 1$  and the left hand endpoint of the largest  $s$  interval containing zero on which a solution exists tends to zero as  $s \rightarrow 0^-$ . One concludes from this that for a standard linear solid,  $G(t)$  converges exponentially to its steady-state limit for  $0 < \gamma < 1$  and  $1 < \gamma < m(0)^{(1/2)}$  but not for  $\gamma = 1$ . Moreover, the rate of exponential convergence, i.e.  $a_{\max}$ , tends to zero as  $\gamma \rightarrow 1 \pm$ .

It is also easy to see from (4.2) that  $a_{\max}$  must vanish as  $r \rightarrow m(0)^{(1/2)}$ .

## 5. Conclusions

The principle contributions of this paper to the study of transient mode III crack propagation are the solutions for the displacement and stresses for general loadings and general shear moduli and the inclusion of a failure zone into the model and the calculation of the energy flux,  $G(t)$ , into the failure zone. It was then observed that significant qualitative and quantitative differences exist in the behavior of  $G(t)$  as a function of time, crack speed, and material properties between a model incorporating a failure zone and one which does not.

The question of the rate of convergence of  $G(t)$  to its steady-state limit as  $t \rightarrow \infty$  was also investigated. It was observed that this rate of convergence depends in a complicated way upon the rate of stress relaxation and crack speed. In particular, for a standard linear solid in which stresses relax exponentially fast,  $G(t)$  converges exponentially fast for all crack speeds except the equilibrium shear wave speed. Thus for crack speeds near the equilibrium shear wave speed, it is expected that steady-state conditions would set in more slowly than for crack speeds above or below it. Also the exponential rate of convergence is lost at the glassy shear wave speed. In contrast, for power law material, an exponential rate of

convergence of  $G(t)$  does not occur for any crack speeds less than or equal to the equilibrium shear wave speed whereas for speeds between the equilibrium and glassy shear wave speeds  $G(t)$  does converge to steady state at an exponential rate.



## Appendix

The behavior of  $\beta$  as  $s \rightarrow \infty$  is determined by equation (4.2). If one lets  $h = \beta/s > 0$  then (4.2) becomes

$$\tilde{m}(s[\tau + \alpha\gamma h]) = \gamma^2 (1 + (\tau/\alpha\gamma h))^2. \quad (A.1)$$

The left-hand side  $\tilde{m}(s[\tau + \alpha\gamma h]) = m(0) + \int_0^\infty e^{-(\tau + \alpha\gamma h)s} r_{m'}(r) dr$

has the limit  $m(0) = \left[\frac{c}{c_*}\right]^2 > \gamma^2$  as  $s \rightarrow \infty$ . Therefore (A.1) will be satisfied only if  $h = h_\infty + o(1)$  where  $h_\infty > 0$ . As  $s \rightarrow \infty$ , (A.1) becomes

$$\left[\frac{c}{c_*}\right]^2 = \gamma^2 (1 + (\tau/\alpha\gamma h_\infty))^2 \quad (A.2)$$

It is easily seen that  $h_\infty = \frac{\tau}{\alpha[(c/c_*) - \gamma]}$  and thus

$$\beta = \frac{\tau s}{\alpha[(c/c_*) - \gamma]} + o(s) \text{ as } s \rightarrow \infty.$$

To find  $\bar{G}(s)$  in terms of powers of  $s^{-1}$ , one needs to determine the expansions of each of the individual factors. First, one needs to find

$$\tilde{m}(\tau s + \alpha\gamma/\epsilon) = m(0) + \int_0^\infty e^{-(\tau s + \alpha\gamma/\epsilon)r} r_{m'}(r) dr = m(0) + m_1 s^{-1} + o(s^{-1}).$$

The  $s^{-1}$  term is found by taking the limit as  $s \rightarrow \infty$  of

$$s \int_0^\infty e^{-(\tau s + \alpha\gamma/\epsilon)r} r_{m'}(r) dr. \text{ After integrating by parts, one}$$

finds that  $m_1 = \frac{1}{\tau} m'(0)$ . Therefore

$$\tilde{m}(\tau s + \alpha\gamma/\epsilon) = m(0) + \frac{1}{\tau} m'(0) s^{-1} + o(s^{-1}) \text{ and}$$

$$[\tilde{m}(\tau s + \alpha\gamma/\epsilon)]^{-1} = \frac{1}{m(0)} \left[ 1 - \frac{m'(0)}{m(0)} \frac{1}{\tau s} \right] + o(s^{-1}). \quad (A.3)$$

The expression  $\frac{|1 - (\epsilon\beta)^2|^{(-1/2)}}{(1+\beta)}$  can be expanded as

$$\epsilon \left[ 1 - \alpha [c/c_* - \gamma] \frac{1}{rs} \right] + o(s^{-1}) \text{ as } s \rightarrow \infty. \quad (\text{A.4})$$

$$\begin{aligned} \text{The final term} & \left| \frac{(\alpha/\epsilon)^2}{(rs + \alpha\gamma/\epsilon)^2} - \frac{1}{\tilde{m}(rs + \alpha\gamma/\epsilon)} \right|^{-(1/2)} \\ &= \left| o(s^{-2}) - \frac{1}{m(0)} \left[ 1 - \frac{m'(0)}{m(0)} \frac{1}{rs} \right] + o(s^{-1}) \right|^{-(1/2)} \\ &= m(0)^{(1/2)} \left[ 1 + \frac{m'(0)}{2m(0)} \frac{1}{rs} \right] + o(s^{-1}). \end{aligned} \quad (\text{A.5})$$

If (A.3)-(A.5) are substituted into (3.19), one obtains (4.4).

To determine the behavior of  $\beta$  as  $s \rightarrow 0$ , one must again consider equation (4.2). Note that the left-hand side of (4.2)

$$\tilde{m}(rs + \alpha\gamma\beta) = m(0) + \int_0^\infty e^{-(rs + \alpha\gamma\beta)r} m'(r) dr \geq 1. \quad (\text{A.6})$$

For  $0 < v < c_*$ , the limit as  $s \rightarrow 0$  of the right-hand side of (4.2),  $\gamma^2(1 + rs/\alpha\gamma\beta)^2$ , will satisfy (A.6) only if  $\beta = \beta_1 s + o(s)$  as  $s \rightarrow 0$ . Therefore as  $s \rightarrow 0$ , (4.2) becomes

$$1 = \gamma^2(1 + r/\alpha\gamma\beta_1)^2 \quad (\text{A.7})$$

The solution of (A.7) for  $\beta_1$  is easily seen to be  $\beta_1 = \frac{\gamma}{\alpha[1-\gamma]}$ .

For  $c_* < v < c$ , the right-hand side of (4.2) satisfies

$$\gamma^2(1 + rs/\alpha\gamma\beta)^2 \geq \gamma^2 > 1. \quad (\text{A.8})$$

If  $\beta \rightarrow 0$  as  $s \rightarrow 0$  then it can be seen from (A.6) that the limit of  $\tilde{m}(rs + \alpha\gamma\beta) = 1$  as  $s \rightarrow 0$  which contradicts (A.8). Thus

$\beta = \beta_0 + o(1)$ ,  $\beta_0 > 0$  as  $s \rightarrow 0$  where  $\beta_0$  satisfies

$$\gamma^2 - (c/c_*)^2 = \int_0^\infty e^{-\alpha\gamma\beta_0 r} m'(r) dr.$$

For  $v = c_*$ , it can be shown that to satisfy (4.2) that  $\beta \rightarrow 0$  and  $s/\beta \rightarrow 0$  as  $s \rightarrow 0$ . Therefore consider  $\beta = \beta_1 s^\eta + o(s^\eta)$ ,

$0 < \eta < 1$ , as  $s \rightarrow 0$ . If one substitutes this into (4.2) and simplifies the equation, one obtains

$$-\alpha\gamma\beta_1 s^\eta \int_0^\infty r m'(r) dr + o(s^\eta) = \frac{2\tau s^{1-\eta}}{\alpha\gamma\beta_1} + o(s^{1-\eta}) \text{ as } s \rightarrow 0 \quad (\text{A.9})$$

assuming that  $\int_0^\infty r m'(r) dr$  exists. If this integral does not exist then one must do the asymptotics for the particular shear modulus needed. Since the coefficient of each side is

non-zero,  $\eta = 1/2$  and  $\beta_1 = \frac{\sqrt{2\tau}}{\alpha\gamma} \left[ -\int_0^\infty r m'(r) dr \right]^{(-1/2)}$ . In summary,

$$\beta = \frac{\tau s}{\alpha[1-\gamma]} + o(s) \text{ for } 0 < v < c_*,$$

$$\beta = \frac{\sqrt{2\tau}}{\alpha\gamma} \left[ -\int_0^\infty r m'(r) dr \right]^{(-1/2)} s^{1/2} + o(s^{1/2}) \text{ for } v = c_*, \text{ and}$$

$$\beta = \beta_0 + o(1) \text{ for } c_* < v < c.$$

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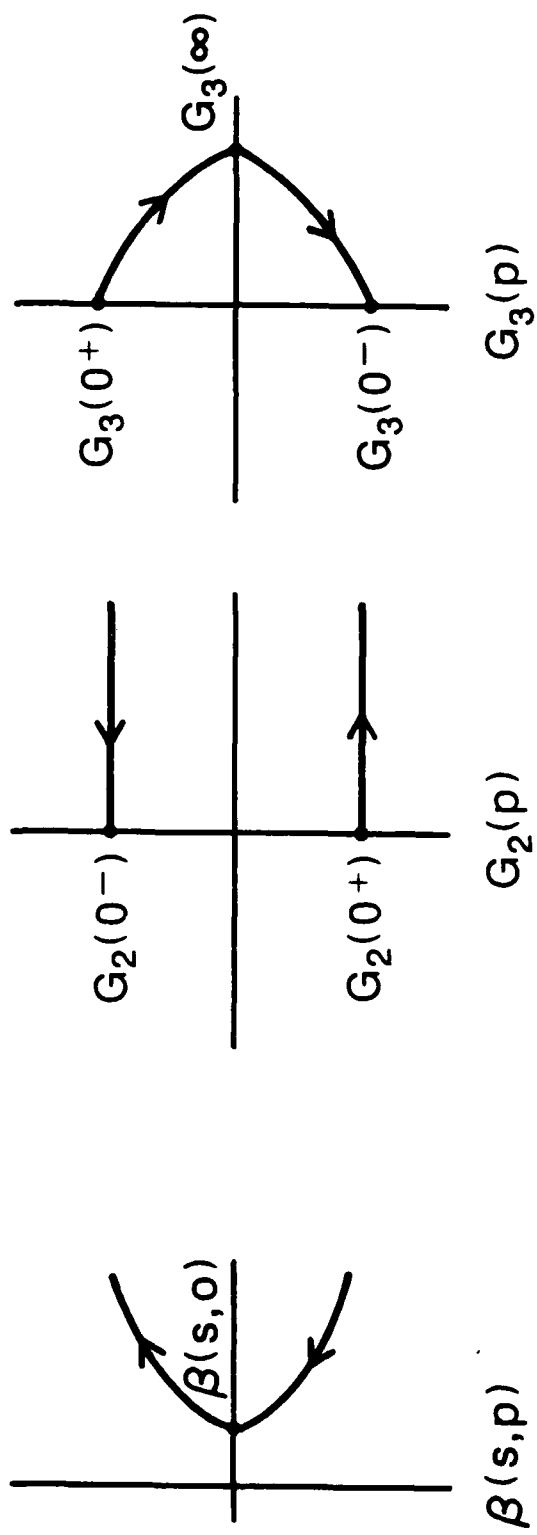
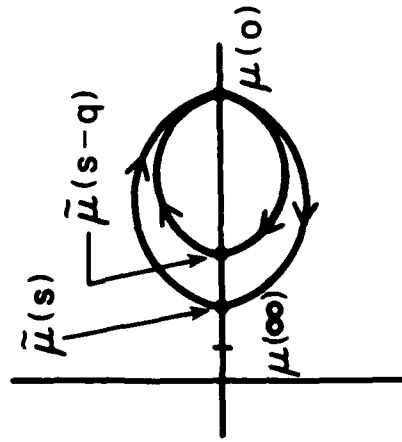
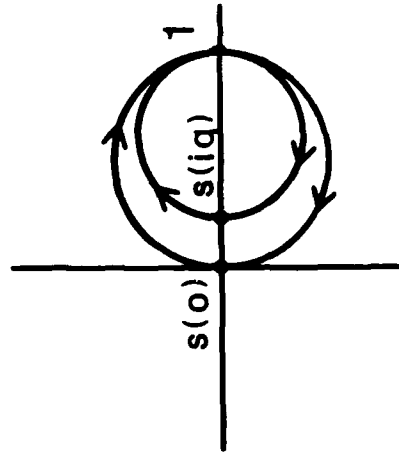


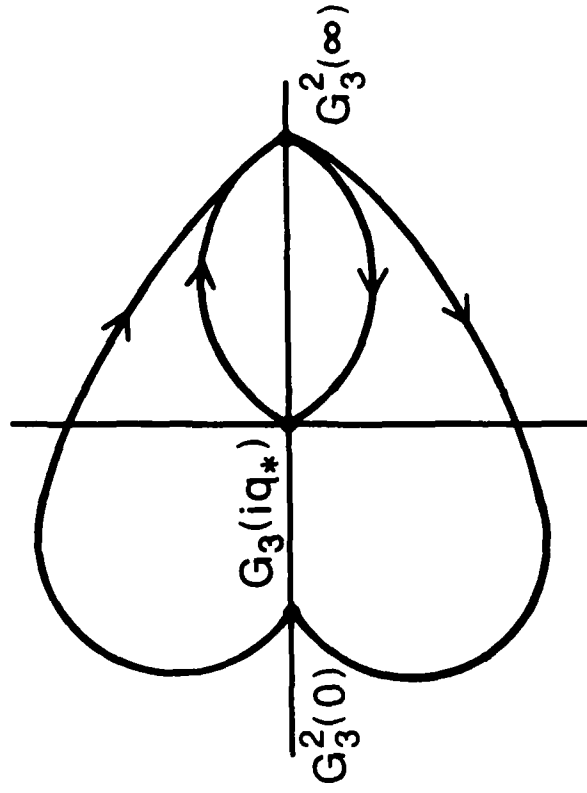
Fig. 2.1



(a)  $\tilde{\mu}(s-q+ivp)$



(b)  $s(p+iq)$



(c)  $G_3^2(p+iq)$

Fig. 2.2

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